

COHOMOLOGY OF INFINITE ALGEBRAS

BY

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1. **Introduction.** If A is an arbitrary algebra over a (commutative) field F , the n th cohomology group $H^n(A, N)$ of A with coefficients in a two-sided A -module N was introduced by Hochschild [8]. In terms of these groups, the (cohomological) dimension of A , $F\text{-dim } A$ (or just $\dim A$ when the base field is understood), is defined as the largest n such that $H^n(A, N) \neq 0$ for some N . In Hochschild's definitions, the 1-cocycles are derivations (over F) of A into N ; the 1-coboundaries are the inner derivations; the 2-cocycles are the factor sets of A into N ; the 2-coboundaries are the split (trivial) factor sets.

The assertion $H^2(A, N) = 0$ then signifies that every algebra extension B of N by A (where multiplication is defined in N by $N^2 = 0$) is cleft: $B = A_1 + N$ (vector space direct sum) with A_1 a subalgebra isomorphic to A . Thus the Wedderburn Principal Theorem can be rephrased in cohomology terms as follows: If A is a separable algebra of finite order, then $\dim A < 2$ (A is absolutely segregated) [8, §6]. (Actually, the Wedderburn Theorem deals with extensions of arbitrary nilpotent algebras N by A , but the proof quickly reduces to the case $N^2 = 0$.)

The assertion $H^1(A, N) = 0$ means every derivation of A into N is inner. It can also be phrased in terms of algebra extensions: If $B = A_1 + N = A_2 + N$ with $N^2 = 0$ and $A_1 \cong A_2 \cong A$, then $A_1 = \sigma(A_2)$ with σ a quasi-inner automorphism⁽²⁾ of B . Hence the Malcev Uniqueness Theorem may be translated thus: If A is a separable algebra of finite order, then $\dim A < 1$. (As in the Wedderburn Theorem, the full Malcev Theorem deals with the case $N^k = 0$, which easily reduces to the case $N^2 = 0$.)

Some of our theorems give necessary, some sufficient, and some necessary and sufficient conditions that certain algebras have dimension n . Any conclusions that assert $n < 2$ (or $n < 1$) can then be interpreted as generalizations of the Wedderburn Theorem (Malcev Theorem) to more general residue-class algebras.

Recently, Cartan and Eilenberg [4] have developed a new approach to cohomology theory, including Hochschild's as a special case. Their work has provided a much more flexible computational technique which has been essential in most of our proofs. In particular, we would like to thank Professor

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⁽²⁾ $\sigma(b) = (1-n)b(1-n)^{-1} = b - nb + bn$ for fixed n in N .

Eilenberg for the use of the galleys of [4] and for several helpful conversations and comments with regard to our paper.

In §2 we give a brief sketch of the techniques we need. The rest of the paper is devoted principally to proving the following results:

1. $\dim A = 0$ if and only if A is a separable algebra of finite order.
2. Let A be a locally finite, semisimple algebra with minimum condition. If A has order \aleph_0 , then $H^1(A, A \otimes A) \neq 0$ (Theorem 3).

3. If A has order \aleph_0 and is locally separable, then $\dim A = 1$ ([13] and Theorem 4).

4. If A is an algebra over K and $K \supset F$ then $F\text{-dim } K \leq F\text{-dim } A \leq F\text{-dim } K + K\text{-dim } A$ (Theorem 5).

5. If A is a field of transcendence degree $n \leq \infty$ over F , then $\dim A \geq n$ (Theorem 8).

6. If A is a field of transcendence degree $n < \infty$ over F , then $\dim A = n$ if and only if A is finitely separably generated (i.e., A is a finite, separable, algebraic extension of a rational function field) (Theorem 10).

7. If A is a finitely generated extension field of F with no separable generation over F , then $\dim A = \infty$ (Theorem 9).

8. Let A be an extension field of transcendence degree n over F which is countably generated but not finitely generated. Then $\dim A = n+1$ or ∞ according as A is or is not locally separably generated⁽³⁾ (Corollary to Theorem 10).

Items 1, 2, and 3 are theorems about algebras of dimensions 0 and 1. Hochschild [8, Theorem 4.1] has already proved that for an algebra A of finite order, $\dim A = 0$ is equivalent to separability. Our contribution to 1 is the fact that if A has infinite order then $\dim A \neq 0$ (Theorem 1), i.e., $H^1(A, N) \neq 0$ for some N . 2, and also Theorem 2, show that in certain cases N may be taken to be $A \otimes A$. It is interesting to notice (Theorem 3) that in case A is semisimple of finite order, this same special module $N = A \otimes A$ has $H^1(A, N) = 0$, regardless of whether or not $\dim A = 0$.

Item 3 provides the major part of the information we have on the dimension of algebraic algebras of infinite order. (Algebras of finite order and dimension > 0 have been treated extensively by Ikeda, Nagao, and Nakayama [11] and by Eilenberg [6].) Miscellaneous extra information may be obtained for example from Theorem 5—which will guarantee that any algebra has dimension ∞ when its center has a subfield of dimension ∞ —and from the results of [11] and [6] coupled with Proposition 3. We should remark that the denumerability hypothesis in 3 cannot be eliminated, for if A is the direct sum of uncountably many copies of F and \bar{A} is the algebra obtained by adjoining a unit to A , then $H^2(\bar{A}, A \otimes A) = H^2(A, A \otimes A) \neq 0$ [14, p. 316]. We conjecture that, at least when the algebra is a field, nondenumerable order implies a dimension > 1 .

⁽³⁾ A field is locally separably generated in case every finite subset can be embedded in a finitely separably generated extension of F .

Item 4 is a useful theorem on subadditivity of dimension. That the dimension is not additive is shown by counterexamples in §4. However, 4 can be easily supplemented to show that the dimensions of the algebras of polynomials and of rational functions in n indeterminates are n . From this we can also derive a new proof that the global dimension of this polynomial ring is n as well.

The remaining theorems concern dimensions of algebras which are fields. As a matter of fact, this also gives equally extensive results on the dimension of commutative algebras with minimum condition, since such an algebra has dimension ∞ unless it is semisimple; but a semisimple commutative algebra is a finite direct sum of fields A_i and has dimension equal to $\max_i \dim A_i$.

2. **Background.** We shall use the following notations consistently: A and K for algebras over a commutative ring F ; A^* for the inverse of A (the algebra anti-isomorphic to A); A° for $A \otimes A^*$, an algebra over F . Tensor products will be understood to be taken over F unless otherwise indicated. All algebras and rings will be assumed to have a unit. A subring shall always contain the unit of the big ring.

Modules over rings will always be unitary, and will be left-modules unless otherwise specified. If R is a ring we shall use R_* for the left R -module R and R_d for the right R -module R . If A is an algebra over F then two-sided A -modules (vector spaces over F having A as both left- and right-operators) are exactly the same as left A° -modules with $am = (a \otimes 1)m$ and $ma = (1 \otimes a^*)m$ for a in A and m in the module. In particular A is a left A° -module in a natural way. So also is $A_* \otimes A_d$; in fact the latter is just A° as a left A° -module, i.e., $(A^\circ)_*$.

The phrases *semisimple* and *simple* shall always mean semisimple and simple with minimum condition on one-sided ideals.

If R is any ring Cartan-Eilenberg [4] define the left dimension of any left R -module M as follows: A *projective resolution* of M is an exact sequence of R -modules and mappings

$$(1) \quad \cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

with X_i a *projective module* (M_0 -module)—a direct summand of a free R -module. (One way to obtain such a projective resolution is to write M as a free R -module X_0 modulo a relations module Y_0 , write $Y_0 = X_1/Y_1$ with X_1 free, etc.) Then the *left dimension* of the R -module M , denoted by $\text{l.dim}_R M$, is the smallest n such that $X_{n+1} = X_{n+2} = \cdots = 0$ for some projective resolution—the length of the shortest projective resolution. It is a nonnegative integer or infinity. The maximum of $\text{l.dim}_R M$ as M ranges over all R -modules is called the *left global dimension* of R and is denoted by $\text{l.gl.dim } R$.

The *dimension* of an algebra A is $\text{l.dim}_{A^\circ} A$ and is denoted by $\dim A$ or $F\text{-dim } A$ when it is desirable to indicate the base ring F (entering into $A^\circ = A \otimes_F A^*$). If F is a field the ordinary dimension of A as a vector space

over F will be referred to as the *order* of A and will be denoted by $(A:F)$.

If M and N are R -modules the group $\text{Ext}_R^n(M, N)$ is defined as the n th (co)homology group of the cochain complex

$$0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(X_0, N) \rightarrow \text{Hom}_R(X_1, N) \rightarrow \cdots$$

where the X 's arise from a projective resolution (1) of M . The fundamental uniqueness of [4, V, 3] asserts that Ext is independent of the particular projective resolution used. Then $\text{l. dim}_R M$ may be defined equivalently as the largest n such that $\text{Ext}_R^n(M, N) \neq 0$ for some N . Since for an algebra A , $\text{Ext}_A^n(A, N) \cong H^n(A, N)$ [4, IX, 6], $\dim A$ defined as $\text{l. dim}_A A$ coincides with the definition in §1.

At several points in the discussion we shall find it necessary to use the Hochschild definitions and terminology rather than those of [4]. We therefore recall briefly what we need: The n -cochains of A into the two-sided A -module N are the F -multilinear functions from A to N and the coboundary operator δ defined by

$$\begin{aligned} \delta f(x_1, \cdots, x_{n+1}) &= x_1 f(x_2, \cdots, x_{n+1}) + \sum (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) \\ &\quad + (-1)^{n+1} f(x_1, \cdots, x_n) x_{n+1} \end{aligned}$$

sends n -cochains into $n+1$ cochains. The n -cochains f such that $\delta f = 0$ are the n -cocycles and the n -cochains of the form δg with g an $(n-1)$ -cochain are the n -coboundaries. Then the additive group of n -cocycles modulo n -coboundaries is $H^n(A, N)$.

3. Infinite algebras.

THEOREM 1. *Let A be an algebra over a commutative ring F which is free as an F -module. (This is automatic if F is a field.) If $\dim A = 0$, then A is finitely F -generated.*

Proof. The following proof was kindly pointed out to us by the referee and replaces our earlier proof which was only valid in case F was a field: Let $\{a_i\}$ be an F -basis of A . Renumbering the basis elements if necessary, [4, IX, 7.7] shows the existence of elements $\bar{a}_1, \cdots, \bar{a}_k$ in A such that

$$(i) \sum a_i \bar{a}_i = 1 \quad \text{and} \quad (ii) \sum a a_i \otimes \bar{a}_i = \sum a_i \otimes \bar{a}_i a,$$

for any a in A (cf. also [9, Theorem 5]). Let B denote the F -module generated by $\bar{a}_1, \cdots, \bar{a}_k$. Then if we write $a a_i$ as an F -linear combination of the a_i , (ii) shows that $\bar{a}_i a$ is an F -linear combination of the \bar{a}_i . Hence B is a right ideal in A . Now, $a = 1 \cdot a = \sum a_i \bar{a}_i a$, by (i), so that $A = \sum a_i B$. Thus A is a finitely generated F -module and so, of course, possesses a finite F -basis.

REMARKS. 1. Using [8, Theorem 4.1] we conclude that if F is a field a necessary and sufficient condition for $\dim A = 0$ is $(A:F) < \infty$ and A separable.

2. Even if A has no unit element, Theorem 1 is true. Only Hochschild's definitions apply to this case but [9, Theorem 2] asserts that one may adjoin a unit to A and get a new algebra \bar{A} of the same dimension. Since A is F -free, $\bar{A} = A \oplus F \cdot 1$ is also and if A is finitely generated as an F -module, \bar{A} will also be.

3. If F is a field, $\dim A = 0$ is equivalent to A^* being semisimple. Hence over a field $A \otimes A^*$ will be semisimple if and only if $(A:F) < \infty$ and A is separable.

If F is a field Theorem 1 states that if $(A:F)$ is infinite, there is an A^* -module N such that $H^1(A, N) \neq 0$. For later applications we need information about H^1 with coefficients in a specific A^* -module. Theorems exhibiting behavior of specific coefficient modules are already known. For example, if A is a semisimple subalgebra of a central simple algebra L of finite order, then every derivation of A into L is inner; i.e. $H^1(A, L) = 0$ [12, Chap. 5, Theorem 17]. Of course, if A is separable, $H^1(A, N) = 0$ for every A^* -module N , but H^1 vanishes for these specific coefficients even if A is inseparable. By reduction to the classical theorem we shall prove a similar result with the A^* -module $A_s \otimes A_d$ as coefficients. It is interesting that $A_s \otimes A_d$ is the same module which can be used for certain algebras of infinite order to give a nonvanishing first cohomology group (Theorems 2 and 3). To investigate $H^1(A, A_s \otimes A_d)$ we first study $H^1(A, M \otimes N)$ where M and N are irreducible left and right A -modules, respectively.

PROPOSITION 1. *Let A be a simple algebra of finite order over a field F , let M and N be irreducible left and right A -modules respectively, and L be the algebra of all F -linear transformations on M . Since L contains a subalgebra isomorphic to A (the operators on M induced by A), L may be considered as a two-sided A -module. As such, $L \cong M \otimes N$. Furthermore, if P and Q are finitely generated left and right A -modules respectively, $H^1(A, P \otimes Q) = 0$.*

Proof. The dual M^* of the F -space M is a right A -module isomorphic to N . To verify this it is sufficient to note that M^* is irreducible, which is an immediate consequence of the fact $(M^*:F) = (M:F)$. Thus $M \otimes N$ is A^* -isomorphic to $M \otimes M^*$. But the classical F -isomorphism $L \cong M \otimes M^*$ which associates to $m_1 \otimes m_2^*$ the mapping $m \rightarrow m_2^*(m)m_1$ is trivially verified to be a two-sided L -isomorphism and so also a two-sided A -isomorphism.

By the theorem mentioned above [12, Chap. 5, Theorem 17],

$$H^1(A, M \otimes N) = H^1(A, L) = 0.$$

Since P is a finite direct sum of modules isomorphic to M and Q is a finite direct sum of modules isomorphic to N , and since $H^n(A, X)$ distributes over finite direct sums in the second variable, $H^1(A, P \otimes Q) = 0$.

PROPOSITION 2. *Let A be a simple algebra of finite order over a field F , let P*

and Q be finitely generated left and right A -modules respectively and let C be the centralizer of A on $P \otimes Q$:

$$C = \{x \in P \otimes Q \mid xa = ax \text{ for all } a \text{ in } A\}.$$

Then $(C:F)(A:F) = (P:F)(Q:F)$.

Proof. If $P=M$ and $Q=N$ are as in Proposition 1, then C is A^* -(hence F -)isomorphic to the centralizer of A in L . The standard theorem on centralizers of simple subalgebras of central simple algebras [1, IV, Theorem 13] then asserts that $(C:F)(A:F) = (L:F) = (M:F)(N:F)$. In general P is a direct sum of p copies of M , Q is a direct sum of q copies of N and hence $P \otimes Q$ is a direct sum of pq copies of $M \otimes N$. The centralizer C of A in $P \otimes Q$ is the direct sum of the centralizers of A in the direct summands. Hence $(C:F)(A:F) = pq(M:F)(N:F) = (P:F)(Q:F)$.

THEOREM 2. *If F is a field and the F -algebra A is a union of a countably infinite tower of simple subalgebras of finite order, then $H^1(A, A_* \otimes A_d) \neq 0$.*

Proof. Write $A = \bigcup_{i=1}^{\infty} A_i$ with $A_1 \subset A_2 \subset \dots$, a tower of simple subalgebras of finite order. Since the tower is properly infinite, we may assume, deleting some of the A_i 's if necessary, that $(A_{i+1}:F) \geq 3(A_i:F)$; this we shall have occasion to do near the end of the proof.

We write $X = A_* \otimes A_d$ and introduce on X a topology with base at zero consisting of the centralizers

$$C_i = \{x \in X \mid xa - ax = 0 \text{ for all } a \in A_i\}.$$

(It can be shown that this is a Hausdorff topology but we have no need for that fact.) The brunt of the proof consists in showing X is not complete. In fact, we shall exhibit a sequence $\{x_i\}$ with $x_i \in C_i$ but with $\sum_{i=1}^{\infty} x_i$ not convergent in X . Then the mapping

$$(2) \quad a \rightarrow \Delta(a) = \sum_{i=1}^{\infty} (x_i a - a x_i)$$

is a non-inner derivation of A into X . This mapping is well-defined because each a in A is in some A_k and since $x_i \in C_i \subset C_k$ for $i \geq k$, all terms in the series (2) beyond the $(k-1)$ st vanish. This also shows that on each A_k the mapping is a derivation; in fact, inner derivation by $y_k = \sum_{i=1}^{k-1} x_i$. Hence Δ is a derivation on A . However, if Δ is inner on A , i.e., $\Delta(a) = xa - ax$ for fixed $x \in X$, then for $a \in A_k$, $xa - ax = y_k a - a y_k$, $x - y_k \in C_k$, so that $x = \lim y_k = \sum x_i$, contrary to the choice of $\{x_i\}$. It thus remains to choose $\{x_i\}$.

Denote by X_i the A_i^* -module $A_{i*} \otimes A_{id}$, which gives $X = \bigcup_{i=1}^{\infty} X_i$. Let U_i and V_i be left and right A_i -complements of A_i in A_{i+1} , respectively. Then

$$X_{i+1} = X_i \oplus (A_{i*} \otimes V_i) \oplus (U_i \otimes A_{id}) \oplus (U_i \otimes V_i)$$

as A_i^* -modules. Denote by C'_i the centralizer $C_i \cap (U_i \otimes V_i)$ of A_i on $U_i \otimes V_i$

and by C'_{i+1} the projection on $U_i \otimes V_i$ of $C_{i+1} \cap X_{i+1}$. Then by Proposition 2,

$$(C'_i:F) = (U_i:F)(V_i:F)/(A_i:F) = [(A_{i+1}:F) - (A_i:F)]^2/(A_i:F)$$

whereas, also by Proposition 2,

$$(C'_{i+1}:F) \leq (C_{i+1} \cap X_{i+1}:F) = (A_{i+1}:F)^2/(A_{i+1}:F) = (A_{i+1}:F).$$

Denoting $(A_{i+1}:F)/(A_i:F)$ by t and assuming $t \geq 3$, we have

$$(C'_i:F)/(C'_{i+1}:F) \geq (t-1)^2/t > 1$$

which proves that there must exist an element x_i in C'_i which is not in C'_{i+1} . This x_i surely satisfies

$$(3) \quad x_i \in C'_i \subset X_{i+1} \cap C_i,$$

$$(4) \quad x_i \notin X_i + C_{i+1}.$$

If now $\sum_{i=1}^{\infty} x_i$ converges to x in X , then for each n and all sufficiently large r (choose one $r \geq n$), C_{n+1} contains $\sum_{i=1}^r x_i - x = x_n + \sum_{i=1}^{n-1} x_i + \sum_{i=n+1}^r x_i - x$, with $\sum_{i=1}^{n-1} x_i \in X_n$ and $\sum_{i=n+1}^r x_i \in C_{n+1}$ by (3). Thus $x_n \in C_{n+1} + X_n - x$ and if we choose n so that $x \in X_n$ we have $x_n \in C_{n+1} + X_n$, a contradiction. This proves Theorem 2.

THEOREM 3. *Let A be a locally finite⁽⁴⁾, semisimple algebra over a field F . If $(A:F) = \aleph_0$ then $H^1(A, A_s \otimes A_d) \neq 0$, but if $(A:F) < \aleph_0$ then $H^1(A, A_s \otimes A_d) = 0$.*

Proof. If $A = \sum_{\oplus} A_i$ is the decomposition of A as a direct sum of simple algebras A_i , then by [4, IX, 5.3], $H^1(A, A_s \otimes A_d) = \sum_{\oplus} H^1(A_i, A_{is} \otimes A_{id})$. Thus we may reduce to the case where A is simple.

In case $(A:F) = \aleph_0$, write $A = T \otimes D$ where T is a total matrix algebra over F and D is a division subalgebra (hence D is also locally finite). Then D is a union $\bigcup_1^{\infty} D_i$ of a countable tower of division subalgebras of finite order (e.g., D_i is the algebra generated by the first i elements in a basis of D over F). Setting $A_i = T \otimes D_i$, we have a countably infinite tower $A_1 \subset A_2 \subset \cdots$ of simple subalgebras of A whose union is A . Theorem 2 is applicable and completes the proof that $H^1(A, A_s \otimes A_d) \neq 0$.

If $(A:F) < \aleph_0$, $H^1(A, A_s \otimes A_d) = 0$ by Proposition 1.

REMARK. There is no loss of generality in Theorems 2 and 3 in assuming that F is a field. For if a simple ring A is an algebra over a commutative ring F , the center of A is a field containing $F \cdot 1$, hence also the quotient field K of $F \cdot 1$. Thus A is also an algebra over K . Moreover $A \otimes A^* = A \otimes_K A^*$. To see this, it is enough to show $k \otimes 1 = 1 \otimes k$ for each k in K . But $k = f_1 f_2^{-1}$ with f_1, f_2 in $F \cdot 1$, so that $k \otimes 1 = f_1 f_2^{-1} \otimes f_2 f_2^{-1} = 1 \otimes k$. Thus the cohomology groups

(4) Every finite subset can be embedded in a subalgebra of finite order.

and the dimension of A over F are the same as over K . This remark could also be used to generalize the results of §5 to algebras over commutative rings instead of over fields.

If F is a field, Theorem 1 asserts that if $(A:F)$ is infinite then $\dim A \geq 1$. In the next section we shall determine the exact dimension of some transcendental algebras; as the final result of the present section we include a theorem due to Kuročkin [13] which shows that certain algebraic algebras have dimension exactly 1.

THEOREM 4. *If A is a locally separable^(b) algebra over a field F and $(A:F) = \aleph_0$, then $\dim A = 1$.*

Proof. The two assumptions about A immediately imply that A contains a denumerable tower of separable subalgebras of finite order $A_1 \subset A_2 \subset A_3 \subset \cdots$ such that $A = \bigcup_1^\infty A_i$. Since $\dim A_i = 0$ Theorem 4 will follow immediately from Theorem 1 and the following proposition which was kindly communicated to us by Professor Eilenberg.

PROPOSITION 3. *Let F be a field, A an F -algebra, and $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ a sequence of subalgebras of A such that $A = \bigcup_1^\infty A_i$. Then*

$$\dim A \leq 1 + \sup_i \dim A_i.$$

Proof. Assume $\dim A_i < n$ for all i , and let f be an $(n+1)$ -cocycle of A into a two-sided A -module, N . We shall define by induction a sequence of n -cochains g_i from A_i to N such that

$$g_{i+1}|A_i = g_i \quad \text{and} \quad \delta g_i = f|A_i.$$

Assume g_i already defined. Since $H^{n+1}(A_{i+1}, N) = 0$ there is a cochain h on A_{i+1} such that

$$\delta h = f|A_{i+1}.$$

Then $\delta g_i = (\delta h)|A_i$ and thus $g_i - h|A_i$ is an n -cocycle on A_i . Since $H^n(A_i, N) = 0$ there is an $(n-1)$ -cochain k on A_i such that

$$\delta k = g_i - h|A_i.$$

Let k' be an arbitrary extension of the cochain k to A_{i+1} , i.e., $k'|A_i = k$. Such an extension exists since A_i is an F -direct summand of A_{i+1} . If we set

$$g_{i+1} = h + \delta k',$$

then

$$g_{i+1}|A_i = h|A_i + \delta k = g_i, \quad \delta g_{i+1} = \delta h = f|A_{i+1}, \text{ as desired.}$$

^(b) Every finite subset can be embedded in a separable subalgebra of finite order.

The sequence of cochains $\{g_i\}$ defines a single cochain g on A such that $\delta g = f$. Thus $H^{n+1}(A, N) = 0$ and $\dim A \leq n$.

4. Additivity of dimension. We begin this section by reading down two of the "change of rings" theorems of [4].

PROPOSITION 4 [4, VI, 4.1.4]. *Let R be a ring with subring S such that R is projective as a left S -module. Then for any R -module M , $\text{l.dim}_R M \geq \text{l.dim}_S M$.*

PROPOSITION 5 [4, VI, 4.1.3]. *Let R be any ring with subring S such that R is projective as a right S -module. Then for any S -module M ,*

$$\text{l.dim}_S M \geq \text{l.dim}_R R \otimes_S M.$$

We shall also need the following proposition which may be verified by direct computation.

PROPOSITION 6. *When f is a function of $n+1$ variables u_1, \dots, u_{n+1} , denote by $\mathfrak{A}_{n+1}f$ the function*

$$\mathfrak{A}_{n+1}f(x_1, \dots, x_{n+1}) = \sum (\text{sgn } \pi) f(x_{\pi(1)}, \dots, x_{\pi(n+1)})$$

the sum ranging over all permutations π of $\{1, \dots, n+1\}$. If g is an n -cochain on a commutative algebra then⁽⁶⁾

$$\begin{aligned} \mathfrak{A}_{n+1}\delta g(x_1, \dots, x_{n+1}) \\ = \sum_{i=1}^{n+1} (-1)^{i-1} [x_i \mathfrak{A}_n g(x_1, \dots, \hat{x}_i, \dots, x_n) - \mathfrak{A}_n g(x_1, \dots, \hat{x}_i, \dots, x_n) x_i]. \end{aligned}$$

We now prove the main theorem of this section.

THEOREM 5. *Let A be an algebra over the commutative ring K such that A is a free K -module (this is automatic if K is a field), and let F be a subring of K . If $F\text{-dim } K = n$ and $K\text{-dim } A = p$, then $n \leq F\text{-dim } A \leq n + p$.*

Proof. Since A is K -free, $K \cong K \cdot 1$ is in the center of A . Therefore, A is a direct sum of copies of K even as a K^e -module so that by [4, VI, 1.2] $\text{l.dim}_{K^e} A = \text{l.dim}_{K^e} K = n$. Now A^e is also a direct sum of copies of K^e . Thus Proposition 4 yields $F\text{-dim } A = \text{l.dim}_{A^e} A \geq \text{l.dim}_{K^e} A = n$.

Similarly by Proposition 5, $\text{l.dim}_{K^e} K \geq \text{l.dim}_{A^e} A^e \otimes_{K^e} K$. But we have the following natural A^e -isomorphisms: $A^e \otimes_{K^e} K = (A_s \otimes A_d) \otimes_{K \otimes K} K \cong A_s \otimes_{F \otimes K} (A_d \otimes_K K) \cong A_s \otimes_{K \otimes A_d}$, where the isomorphism between the second and third terms is given by $(a \otimes a^*) \otimes k \rightarrow a \otimes (a^* \otimes k)$. We omit the straightforward proof (cf. [4, IX, 2.1]). Thus $\text{l.dim}_{A^e} A_s \otimes_K A_d \leq n$. Since $p = \text{l.dim}_{A \otimes_K A^e} A$ and $A \otimes_K A^*$ is a homomorph of A^e , [7, Proposition 3] is available and asserts $\text{l.dim}_{A^e} A \leq n + p$, proving the theorem.

To show that $F\text{-dim } A$ may very well be less than $n + p$, we offer the following two examples.

⁽⁶⁾ \hat{x}_i , means that x_i is to be omitted.

(i) Let A be a separable algebraic extension field of a field F with $(A:F) = \aleph_0$. Then there is a subfield K of A such that $(K:F) = (A:K) = \aleph_0$ also. Theorem 4 then yields $F\text{-dim } A = K\text{-dim } A = F\text{-dim } K = 1$.

(ii) Let F be a field of characteristic $p \neq 0$, t an indeterminate over F , $K = F(t)$, and $A = K(t^{1/p})$. Then Theorem 7 below shows that $F\text{-dim } K = 1$ and by [10, p. 946] or our Theorem 9, $K\text{-dim } A = \infty$. However, $A = F(t^{1/p})$ is merely a rational function field over F and so again by Theorem 7, $F\text{-dim } A = 1$.

Examples where the dimension is additive are given by the following two theorems.

THEOREM 6 [4, IX, 7.11]. *Let F be a commutative ring, and A be the polynomial ring $F[t_1, \dots, t_n]$ in n independent indeterminates over F . Then $F\text{-dim } A = n$.*

Proof. First, $\dim A \geq n$ because we can exhibit a nonzero element of $H^n(A, A)$: with u_1, \dots, u_n varying in A , let

$$f(u_1, \dots, u_n) = \prod_{i=1}^n \frac{\partial u_i}{\partial t_i}$$

where $\partial u_i / \partial t_i$ denotes the ordinary partial derivative of the polynomial u_i . Routine calculation shows that this is an n -cocycle of A into A . Since A is commutative, any coboundary δg of A into A satisfies $\mathfrak{A}_n \delta g(u_1, \dots, u_n) = 0$ by Proposition 6. But direct evaluation gives $\mathfrak{A}_n f(t_1, \dots, t_n) = 1$, so that f is not a coboundary. (Essentially the same proof that f is not a coboundary is already to be found in [5].)

Next we prove $\dim A \leq n$ by induction on n . If $n = 1$, $A = F[t]$, $A^e = A \otimes A^* = F[t', t'']$, where $t' = t \otimes 1$ and $t'' = 1 \otimes t$ are independent indeterminates over F . The natural ring-homomorphism of $A^e = A \otimes A^*$ onto $A = A \otimes_A A^*$ has as kernel the principal ideal generated by $t' - t''$ since $p(t', t'')$ is in the kernel if and only if $p(t, t) = 0$. But the only element in A^e annihilating $t' - t''$ is 0 so this kernel is a free A^e -module and

$$0 \rightarrow (t' - t'')A^e \rightarrow A^e \rightarrow A \rightarrow 0$$

is a projective resolution for A over A^e . Thus $\dim A \leq 1$. Now assume the inequality true for the algebra $K = F[t_1, \dots, t_{n-1}]$ and consider $A = K[t_n]$. By the case $n = 1$, $K\text{-dim } A \leq 1$ and by the induction hypothesis, $F\text{-dim } K \leq n - 1$. Hence Theorem 5 proves $F\text{-dim } A \leq n$.

THEOREM 7. *Let F be a field and A the field of rational functions*

$$F(t_1, \dots, t_n)$$

in n independent indeterminates over F . Then $F\text{-dim } A = n$.

The proof is identical with that of Theorem 6 once we note that when

$n=1$, A° is the integral domain consisting of all rational functions

$$p(t', t'')/q(t')r(t'')$$

with p, q, r polynomials over F .

Besides Theorem 6, Cartan and Eilenberg also prove that if F is semisimple the global dimension of the polynomial ring A is n . Both these facts are proved in [4] by explicit construction of a projective resolution for F as an A -module. We shall now give an alternative proof of this result on $\text{l.gl.dim } A$ which reduces the problem to one of algebra dimension and avoids this projective resolution.

If F is semisimple, $\text{l.gl.dim } A \leq \dim A$ [4, IX, 7.6]. Since $\dim A = n$ by Theorem 6, it is sufficient to exhibit a left A -module with left A -dimension at least n . We turn F into an A -module by setting $t_i F = 0$. Now, A° is the ring of polynomials in $2n$ indeterminates t'_i, t''_i ($i=1, \dots, n$) with $t'_i = t_i \otimes 1$ and $t''_i = 1 \otimes t_i$. We make it an A -module by identifying A with the subring $A_0 = F[t'_1 - t''_1, \dots, t'_n - t''_n]$ of A° . Then A° is a ring of polynomials in n indeterminates over A_0 and so is a free A_0 -(hence A -)module. Proposition 5 then gives $\text{l.dim}_A F \geq \text{l.dim}_{A^\circ} A^\circ \otimes_A F$. But $A^\circ \otimes_A F \cong A$ as A° -modules, the isomorphism being given by $b \otimes f \rightarrow bf$ where $b \rightarrow bf$ is the natural mapping of A° onto A (⁷). We omit the straightforward verification that this is an A° -isomorphism. Thus $\text{l.dim}_A F \geq \text{l.dim}_{A^\circ} A = \dim A = n$ by Theorem 6, which proves $\text{l.gl.dim } A \geq n$.

5. Commutative algebras with minimum condition. If A is a commutative algebra over a field F satisfying the minimum condition, $\dim A = \infty$ if A is not semisimple ([3, Proposition 15] and [4, IX, 7.6]). Thus we can restrict ourselves to the semisimple case where A is a direct sum of fields. By [4, IX, 7.3] we may then restrict ourselves to the case where A is itself a field.

THEOREM 8. *If A is a field of transcendence degree $n \leq \infty$ over the field F , then $\dim A \geq n$.*

Proof. For any finite $m \leq n$, A contains a rational function field K in m indeterminates. Theorems 5 and 7 then show $m \leq \dim A$.

THEOREM 9. *If A is a finitely generated extension field of F with no separable generation over F , then $\dim A = \infty$.*

Proof. Of course the base field F has a characteristic $p \neq 0$. Let $A = F(x_1, \dots, x_r)$ and let s be the largest integer ≥ 0 such that $S = F(x_1, \dots, x_s)$ can be separably generated over F . Let $\{t_1, \dots, t_n\}$ be a transcendence

(⁷) The tempting string of natural isomorphisms (as in Theorem 5) $A^\circ \otimes_A F = (A \otimes_A A) \otimes_A F \cong A \otimes (A \otimes_A F) \cong A \otimes_F F \cong A$, will not do here. It can be arranged to give a group isomorphism, but if the isomorphisms are all to be A° -isomorphisms the final A is not the usual A° -module A but rather is annihilated by multiplication by $1 \otimes t_i$.

basis of S such that S is separable over the rational function field $K = F(t_1, \dots, t_n)$, and write $y = x_{s+1}$. Then we assert that $K(y)$ has no separable generation over F . For suppose otherwise. Since S is separable over K , $S(y)$ is separable over $K(y)$, whence $F(x_1, \dots, x_s, x_{s+1}) = S(y)$ is separably generated over F , contradicting the choice of s . By Theorem 5, it suffices to prove $\dim K(y) = \infty$.

Let ϕ be an irreducible polynomial in $n+1$ indeterminates over F such that

$$(5) \quad \phi(t_1, \dots, t_n, y) = 0.$$

Since y is inseparable over K and (5) is the minimum equation of y over K , in every nonzero term of ϕ the exponent of y is a multiple of p . If t_i actually occurs in (5) then $\{t_1, \dots, t_i, \dots, t_n, y\}$ ⁽⁶⁾ is another transcendence basis of $K(y)$ over F . Since $K(y)$ has no separable generation over F , t_i plays the same role with respect to $F(t_1, \dots, \hat{t}_i, \dots, t_n, y)$ as y does with respect to K , and the exponents of t_i in (5) are also multiples of p .

Now let G be the field obtained by adjoining to F the p th roots of the coefficients of ϕ so that ϕ becomes the p th power of a polynomial $\phi^{1/p}$ with coefficients in G . Let $z = \phi^{1/p}(t_1, \dots, t_n, y) \in G \otimes K(y)$ so that $z^p = 0$. On the other hand, $z \neq 0$, for if a collection of products of powers of t_1, \dots, t_n, y (as in the terms of $\phi^{1/p}$) involve no higher power of y than appears in $\phi(t_1, \dots, t_n, y)$, then these power products are linearly independent in $K(y)$ over F , hence also in $G \otimes K(y)$ over G . Therefore $G \otimes K(y)$ is a commutative algebra over G satisfying the minimum condition [2, Theorem 6.10B] and containing a nilpotent ideal. By the first sentence in this section,

$$G\text{-dim}(G \otimes K(y)) = \infty.$$

Since the dimension of an algebra is invariant under change of base field [4, IX, 7.2], $F\text{-dim } K(y) = \infty$ also.

We note that Theorem 9 also yields Hochschild's result that a semisimple inseparable algebra of finite order has dimension ∞ [10, p. 946]. Clearly it is sufficient to treat the case of a simple algebra A whose center Z is an inseparable finite extension field of F . Then by Theorem 9, $\dim Z = \infty$ and since A is Z -free, Theorem 5 gives $\dim A = \infty$ also.

THEOREM 10. *If A is a field of transcendence degree $n < \infty$ over a field F , then $\dim A = n$ if and only if A is finitely, separably generated over F (i.e., A is a finite, separable, algebraic extension of a rational function field in n indeterminates).*

Proof. If A is a finite, separable, algebraic extension of a rational function field, K , then $K\text{-dim } A = 0$ [8, Theorem 4.1] and $F\text{-dim } K = n$ by Theorem 7. The sufficiency follows from Theorem 5. Theorem 9 yields one part of the necessity proof. It remains to show that if A is an infinite algebraic extension

of a rational function field $K = F(t_1, \dots, t_n)$ then $\dim A > n$. First, A contains a subfield A' with $(A':K) = \aleph_0$ and by Theorem 5 it is sufficient to prove that $\dim A' > n$. So, without loss of generality, we may assume that $(A:K) = \aleph_0$. Theorem 3 guarantees a K -linear derivation Δ of A into $A_s \otimes_K A_d$ which is not inner. Also K has n derivations $\Delta_i = \partial/\partial t_i$ into itself satisfying $\Delta_i(t_j) = 0$ for $i \neq j$ and $\Delta_i(t_i) = 1$. These Δ_i can be extended to derivations of A thus: In terms of a basis⁽⁸⁾ of A over K left multiplication by an element a in A may be represented as a column-finite matrix μ with entries in K . Define $\Delta_i(a)$ as the matrix obtained by applying Δ_i to all the entries in μ . This gives n derivations of A into the ring of all K -linear transformations on A , which is naturally isomorphic to the ring of all $(1 \otimes_K A)$ -linear transformations on $A_s \otimes_K A_d$. Since the elements of K are represented by scalar matrices, we still have $\Delta_i(t_j) = 0$ for $i \neq j$ and $\Delta_i(t_i) = 1$. Then construct

$$(6) \quad f(u_1, \dots, u_{n+1}) = \Delta_1(u_1)\Delta_2(u_2) \cdots \Delta_n(u_n)\Delta(u_{n+1}).$$

Direct computation shows this is an $(n+1)$ -cocycle of A into $A_s \otimes_K A_d$. Suppose $f = \delta g$ and use the identity in Proposition 6 with $u_i = t_i$ ($i = 1, \dots, n$) and u_{n+1} arbitrary. The left side of the identity becomes $\Delta(u_{n+1})$ because $\Delta_i(t_j) = 0$ for $j \neq i$. Since $u_i = t_i \in K$, $u_i x - x u_i = 0$ for $x \in A_s \otimes_K A_d$ and the right side of the identity becomes $u_{n+1}y - y u_{n+1}$ with $y = \mathfrak{A}_n g(t_1, \dots, t_n)$. This means Δ is inner, contrary to construction (Theorem 3). Thus f determines a nonzero element of $H^{n+1}(A, A_s \otimes_K A_d)$, proving Theorem 10.

If A is separable over K , the proof can avoid the assumption $(A:K) = \aleph_0$, the use of Theorem 3 and the matrix arguments: extend the partial derivatives Δ_i on K in the usual way to derivations of A into A . Let Δ be any noncobounding 1-cocycle of A over K into an $(A \otimes_K A^*)$ -module N (Δ exists by Theorem 1). Then construct the noncobounding $(n+1)$ -cocycle f of A into N by the formula (6).

In case $(A:K) = \aleph_0$ we can actually determine $\dim A$ completely:

COROLLARY. *Let A be an extension field of transcendence degree n over F which is countably generated but not finitely generated. Then $\dim A = \infty$ or $n+1$ according as A is not or is locally separably generated⁽³⁾.*

Proof. If A is not locally separably generated, Theorem 9 and Theorem 5 assert $\dim A = \infty$. (This of course does not need the countable generation.) Otherwise $A = \bigcup_1^\infty A_i$ where each A_i is a finitely separably generated extension of F and so has dimension $\leq n$ by Theorem 10. Proposition 3 then shows that $\dim A \leq n+1$ and Theorem 10 gives $\dim A \geq n+1$.

⁽⁸⁾ This extension of derivations need not depend on a choice of basis or a matrix representation: Indeed, in a forthcoming paper we shall show that if L is the ring of all linear transformations on a vector space over a division ring K , and F is a subfield of the center of K , any derivation over F of K into K has an extension to a derivation of L into L ; this extension is unique up to an inner derivation. Moreover, $H^n(L, L) \cong H^n(K, K)$.

We note that A may be locally separably generated without being separably generated itself: let F be a perfect field of characteristic $p \neq 0$ and let $A = F(t, t^{p^{-1}}, t^{p^{-2}}, \dots, t^{p^{-n}}, \dots)$. Then $A = \bigcup F(t^{p^{-n}})$ but A is not separably generated. Indeed, if it were it would admit nonzero derivations into itself, but since every element of A is a p th power this is impossible.

Finally we remark that the proof of Theorem 10 even gives some information about noncommutative algebras A . In particular if A contains in its center a rational function field K in n indeterminates over F and if

$$H^1(A, A_* \otimes_K A_d) \neq 0$$

(cf. Theorems 2 and 3) then $F\text{-dim } A > n$. If in addition $(A:K) = \aleph_0$ and A is locally separable over K , then $F\text{-dim } A = n + 1$ —which, when $n = 0$, does not quite include Theorem 4.

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